Weak reflection at the successor of singular

Mirna Džamonja
School of Mathematics
University of East Anglia
Norwich, NR47TJ, UK
M.Dzamonja@uea.ac.uk
http://www.mth.uea.ac.uk/people/md.html
Saharon Shelah
Mathematics Department
Hebrew University of Jerusalem
91904 Givat Ram, Israel
and
Rutgers University
New Brunswick, New Jersey
USA
shelah@sunset.huji.ac.il
http://www.math.rutgers.edu/∼ shelarch

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Abstract

The notion of stationary reflection is one of the most important notions of combinatorial set theory. We investigate weak reflection, which is, as the name suggests, a weak version of stationary reflection. This sort of reflection was introduced in [DjSh 545], where it was shown that weak reflection has applications to various guessing principles, in the sense that if there is no weak reflection, than a guessing principle holds, and an application dealing with the saturation of

normal filters. Further investigations of weak reflection were carried in [CuDjSh 571] and [CuSh 596]. While various ZFC restrictions on the one hand, and independence results on the other, were discovered about the weak reflection, the question of the relative consistency of the existence of a regular cardinal κ such that the first cardinal weakly reflecting at κ is a successor of singular, remained open. This paper answers that question by proving that (modulo large cardinal assumptions close to 2-hugeness) that there indeed can be such a cardinal κ. [1]

0 Introduction and the statement of the results.

Stationary reflection is a compactness phenomenon in the context of stationary sets. To motivate its investigation, let us consider first the situation of a regular uncountable cardinal κ and a closed unbounded subset C of κ. For every of κ many limit points α of C, we have that $C \cap \alpha$ is closed unbounded in α. Now let us ask the same question, but starting with a set S which is stationary, not necessarily club, in κ. Is there necessarily $\alpha < \kappa$ such that $S \cap \alpha$ is stationary in α-in the lingo of set theorists, S reflects at α? The answer to this question turns to be very intricate, and in fact the notion of stationary reflection is one of the most studied notions of combinatorial set theory. This is the case not only because of the historical significance stationary reflection achieved through by now classical work of R. Jensen [Je] and later work of J.E. Baumgartner [Ba], L. Harrington and S. Shelah [HaSh 99], M. Magidor [Ma] and many later papers, but also because of the large number of applications it has in set theory and allied areas. In set theory, stationary reflection is known to have deep connections with various guessing principles, the simplest one of which is Jensen's $\square$ ([Je]), and the notions from pcf theory, such

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as good scales (for a long list of results in this area, as well as an excellent list of references, we refer the reader to [CuFoMa]), and some connections with saturation of normal filters ([DjSh 545]). In set-theoretic topology, various kinds of spaces have been constructed from the assumption of the existence of a non-reflecting stationary set (for references see [KuVa]), and in model theory versions of stationary reflection have been shown to have a connection with decidability of monadic second-order logic ([Sh 80]).

We investigate the notion of weak reflection, which, as the name suggests, is a weakening of the stationary reflection. For a regular cardinal κ, we say that $\lambda > \kappa$ weakly reflects at κ iff for every function $f : \lambda \to \kappa$, there is $\delta < \lambda$ of cofinality κ (we say $\delta \in S^\lambda_\kappa$) such that $f \restriction e$ is not strictly increasing for any e a club of δ. Its negation is a strong form of non-reflection, called strong non-reflection. The notions were introduced by Džamonja and Shelah in [DjSh 545] in connection with saturation of normal filters, as well as the guessing principle $\clubsuit^*_{-\lambda}(\lambda^+)$, which a relative of another popular guessing principle, $\clubsuit$. It is proved in [DjSh 545] that, in the case when $\lambda = \mu^+$ and $\aleph_0 < \kappa = \mathrm{cf}(\mu) < \mu$, <u>if</u> weak reflection of λ at κ holds relativized to every stationary subset of S^λ_κ, <u>then</u> $\clubsuit^*_{-\mu}(S^\lambda_\kappa)$ holds. The exact statement of the principle is of no consequence to us here, so we omit the definition. We simply note that this statement is stronger than just $\clubsuit^*_{-\mu}(\lambda)$, which holds just from the given cardinal assumptions.

Weak reflection was further investigated by Cummings, Džamonja and Shelah in [CuDjSh 571], more about which will be mentioned in a moment. Weak reflection has a very interesting aplication given by Cummings and Shelah in [CuSh 596], where they use it as a tool to build models where stationary reflection holds for some cofinalities but fails badly for others.

It was proved in [DjSh 545] that if there is λ which weakly reflects at κ, the first such λ is a regular cardinal. It is also not difficult to see that the first λ cannot be weakly compact. On the other hand, in [CuDjSh 571] Cummings, Džamonja and Shelah proved that, modulo the existence of certain large cardinals, it is consistent to have a cardinal λ which weakly reflects at unboundedly many regular κ below it, and strongly non-reflects at unboundedly many others.

A question left open by these investigations, was if it is consistent to have κ for which the first λ which weakly reflects at κ, is a successor of singular. We answer this question positively, modulo the existence of a certain large cardinal, whose strength is in the neighborhood of being 2-huge. To state our results more precisely, let us give the exact definition of weak reflection and the statement of our main theorem.

Definition 0.1. Given $\aleph_0 < \kappa = \mathrm{cf}(\kappa)$ and $\lambda > \kappa$. We say that λ *weakly reflects at* κ iff for every function $f : \lambda \to \kappa$, there is $\delta \in S^\lambda_\kappa$ such that $f \restriction e$ is not strictly increasing for any e a club of δ.

Theorem 0.2. (1) Let V be a universe in which GCH holds and μ_0 is a cardinal such that there is an elementary embedding $\mathbf{j} : V \to M$ with the following properties:

(i) $\mathrm{crit}(\mathbf{j}) = \mu_0$,

(ii) For some κ^* a successor of singular and χ, we have

$$\mu_0 < \kappa^* < \mu_1 \stackrel{\mathrm{def}}{=} \mathbf{j}(\mu_0) < \lambda^* \stackrel{\mathrm{def}}{=} \mathbf{j}(\kappa^*) < \mathrm{cf}(\chi) = \chi < \mu_2 \stackrel{\mathrm{def}}{=} \mathbf{j}(\mu_1),$$

(iii) ${}^\chi M \subseteq M$.

Then there is a generic extension of V in which cardinals and cofinalities $\geq \mu_0$ are preserved, and the first λ weakly reflecting at κ^* is λ^* (hence, a successor of singular).

(2) In (1), we can replace the requirement that κ^* is a successor of singular by "$\varphi(\kappa^*)$ holds" for any of the following meanings of $\varphi(x)$:

(a) x is inaccessible,

(b) x is strongly inaccessible,

(c) x is Mahlo,

(d) x is strongly Mahlo,

(e) x is α-(strongly) inaccessible for $\alpha < x$,

(e) x is α-(strongly) Mahlo for $\alpha < x$,

and have the same conclusion (hence in place of λ^* is a successor of singular, V^P will satisfy $\varphi(\lambda^*)$).
(3) With the same assumptions as in (1),

(i) there is a generic extension of V in which $\kappa^* = \aleph_{53}$ and λ^*, a successor of singular, is the first cardinal weakly reflecting at κ^*,

(ii) there is a generic extension of V in which $\kappa^* = \aleph_{\omega+1}$ and λ^*, a successor of singular, is the first cardinal weakly reflecting at κ^*,

(iii) there is a generic extension of V in which λ^* is the first cardinal weakly reflecting at κ^*, and $\lambda^* = (\kappa^*)^{+\gamma}$ for some $\gamma \le \aleph_7$.

Remark 0.3. Our assumptions follow if μ_0 is 2-huge and GCH holds. The integers 53 and 7 in the statement of part (3) above, are to a large extent arbitrary.

The proof of (1) uses as a building block a forcing notion introduced by Cummings, Džamonja and Shelah in [CuDjSh 571], which introduces a function witnessing strong non-reflection of a given cardinal λ to a cardinal κ. An important feature of this forcing is that it has a reasonable degree of (strategic) closure, provided that strong non-reflection of θ to κ already holds for $\theta \in [\kappa, \lambda)$, and hence it can be iterated. This forcing is a rather homogeneous forcing, so the term forcing associated with it has strong decision properties. The forcing that we actually use is a term forcing associated with a certain product of the strong non-reflection forcings and a Laver-like preparation. Using this, we force the strong non-reflection of θ to κ^* for all $\theta < \lambda^*$, and the point is to prove that in the extension λ^* weakly reflects on κ^*. If we are given a condition and a name forced to be a strogly non-reflecting function, we can use the large cardinal assumptions to pick a certain model N, for which are able to build a generic condition, whose existence contradicts the

choice of the name. To build the generic condition we use the preparation and the fact that we are dealing with a term forcing. Proofs of (2) and (3) are easy modifications of the proof of (1).

We recall some facts and definitions.

Notation 0.4. (1) Reg stands for the class of regular cardinals.
(2) If p, q are elements of a forcing notion, then $p \leq q$ means that q is an extension of p.
(3) For p a condition in the limit of an iteration $\langle P_\alpha, \underset{\sim}{Q}_\beta : \alpha \leq \alpha^*, \beta < \alpha^* \rangle$, we let

$$\mathrm{Dom}(p) \stackrel{\mathrm{def}}{=} \{\beta < \alpha^* : \neg(p \restriction \beta \Vdash \text{“}p(\beta) = \emptyset_{\underset{\sim}{Q}_\beta}\text{”})\}.$$

(4) The statement that λ weakly reflects at κ is denoted by $WR(\lambda, \kappa)$. Its negation of $WR(\lambda, \kappa)$ is denoted by $SNR(\lambda, \kappa)$.

Remark 0.5. It is easily seen that λ weakly reflects at κ iff $|\lambda|$ does, so we can without loss of generality, when discussing weak reflection of λ to κ assume that λ is a cardinal.

Definition 0.6. (1) For a forcing notion and a limit ordinal ε, we define the game $G(P, \alpha)$ as follows. The game is played between I and II, and it lasts ε steps, unless a player is forced to stop before that time. For $\zeta < \varepsilon$, we denote the ζ-th move of I by p_ζ, and that of II by q_ζ. The requirements are that I commences by $\emptyset_P$ and that for all ζ we have $p_\zeta \leq q_\zeta$, while for $\xi < \zeta$ we have $q_\xi \leq p_\zeta$.

I wins a play Γ of $G(P, \varepsilon)$ iff Γ lasts ε steps.
(2) For P and ε as above, we say that P is *ε-strategically closed* iff I has a winning strategy in $G(P, \varepsilon)$. We say that P is *$(< \varepsilon)$-strategically closed* iff it is ζ-strategically closed for all $\zeta < \varepsilon$.

Definition 0.7. A set A of ordinals is an *Easton set* iff

$$\sigma \in \mathrm{Reg} \cap (\sup(A) + 1) \Longrightarrow \sup(A \cap \sigma) < \sigma.$$

Definition 0.8. We shall call a forcing notion P *mildly homogeneous* iff for every formula $\varphi(x_0, \ldots, x_{n-1})$ of the forcing language of P and $a_0, \ldots, a_{n-1}$ (canonical names of) objects in V, we have $\emptyset_P \parallel \text{“}\varphi(a_0, \ldots, a_{n-1})\text{”}$.

1 Proofs.

We give the proof of Theorem 0.2. The main part of the proof is to deal with part (1) of the Theorem, and at the very end of the section we indicate the changes needed for the other parts of the theorem.

Definition 1.1. Given $\aleph_0 < \kappa = \mathrm{cf}(\kappa) < \sigma$.

$\mathbb{P}(\kappa,\sigma)$ is the forcing notion whose elements are functions p with $\mathrm{dom}(p)$ an ordinal $< \sigma$, the range $\mathrm{rge}(p) \subseteq \kappa$, and the property

$$\beta \in S^\sigma_\kappa \Longrightarrow (\exists c \text{ a club of } \beta)\, [p \restriction c \text{ is strictly increasing}],$$

while $\mathbb{P}(\kappa,\sigma)$ is ordered by extension.

Fact 1.2 (Cummings, Džamonja and Shelah). [CuDjSh 571] Let κ and σ be such that $\mathbb{P}(\kappa,\sigma)$ is defined, <u>then</u>

(1) $|\mathbb{P}(\kappa,\sigma)| \le |{}^{<\sigma}\kappa| = \kappa^{<\sigma}$.

(2) Suppose that for all $\theta \in [\kappa,\sigma)$ we have $SNR(\theta,\kappa)$. <u>Then</u> $\mathbb{P}(\kappa,\sigma)$ is $(<\sigma)$-strategically closed.

Definition 1.3. Given $\aleph_0 < \mathrm{cf}(\kappa) = \kappa < \lambda$.

$Q_{(\kappa,\lambda)}$ is the result of the reverse Easton support iteration of $\mathbb{P}(\kappa,\sigma)$ for $\sigma = \mathrm{cf}(\sigma) \in (\kappa,\lambda)$. More precisely, let

$$\bar{Q} = \langle Q_\alpha, \underset{\sim}{R}_\beta : \alpha \le \lambda, \beta < \lambda \rangle,$$

where

(1) $Q_\alpha \Vdash$ "$\underset{\sim}{R}_\alpha = \{\emptyset\}$" unless $\alpha \in \mathrm{Reg} \cap (\kappa,\lambda)$, in which case

$$Q_\alpha \Vdash \text{“}\underset{\sim}{R}_\alpha = \underset{\sim}{\mathbb{P}}(\kappa,\alpha)\text{”}.$$

(2) For $\alpha \le \lambda$

$p \in Q_\alpha$ iff for all $\gamma < \alpha$ we have $\Vdash_{Q_\gamma}$ "$p(\gamma) \in \underset{\sim}{R}_\gamma$" <u>and</u>

(i) if α is inaccessible, then $|\mathrm{Dom}(p)| < \alpha$.

(ii) If α is a limit but not inaccessible, then

$$Q_\alpha \stackrel{\text{def}}{=} \{p : (\forall \beta < \alpha)\, [p \restriction \beta \in Q_\beta]\}.$$

(3) $p \leq q$ iff for all $\beta < \lambda$ we have $q \restriction \beta \Vdash$ "$q(\beta) \geq p(\beta)$".

Fact 1.4 (Cummings, Džamonja, Shelah). [CuDjSh 571] Let $\bar{Q}$, κ and λ be as in Definition 1.3. For all $\alpha \leq \lambda$:

(1) Whenever α is regular, $|Q_\alpha| \leq \alpha^{<\alpha}$,

(2) $\Vdash_{Q_\alpha}$ "$|\underset{\sim}{R}_\alpha| \leq |\alpha|^{<\kappa}$",

(3) $Q_{\alpha+1}$ has $(|\alpha|^{|<\alpha|})^+$-cc. In addition, if α is strongly Mahlo, then Q_α has α-cc.

(4) $\Vdash_{Q_\alpha}$ "$\underset{\sim}{R}_\alpha$ is $(<\alpha)$-strategically closed".

(5) For all $\beta < \alpha$, we have that $\underset{\sim}{Q}_\alpha / Q_\beta$ is $(<\beta)$-strategically closed.

(6) Q_α preserves all cardinals and cofinalities $\geq (|\alpha|^{<|\alpha|})^+$, and all strongly inaccessible cardinals and cofinalities $\leq |\alpha|$.

(7) $\Vdash_{Q_\alpha}$ "$SNR(\kappa, \beta)$" for all $\beta < \alpha$.

Notation 1.5. (1) For a forcing notion Q of the form $Q = P_1 * \underset{\sim}{P}_2$, we denote by $Q^\otimes$ the term forcing associated with Q, defined by

$$Q^\otimes \stackrel{\text{def}}{=} \{(\emptyset_{P_1}, \underset{\sim}{q}) : \underset{\sim}{q} \text{ is a canonical } P_1\text{-name for a condition in } P_2\},$$

(in particular $Q^\otimes \subseteq Q$), with the order inherited from Q.

(2) For a triple (R, κ, σ) with $\aleph_0 < \mathrm{cf}(\kappa) = \kappa < \sigma$, and R a forcing notion preserving $\kappa = \mathrm{cf}(\kappa) > \aleph_0$, with $\emptyset_R$ the minimal element of R, we define $Q^\otimes_{(R,\kappa,\sigma)}$ to be $[R * \underset{\sim}{Q}_{(\kappa,\sigma)}]^\otimes$.

Observation 1.6. $Q^{\otimes}_{(R,\kappa,\sigma)}$, when defined, is $(<\kappa^+)$-strategically closed.

Claim 1.7. (1) $\mathbb{P}(\sigma,\lambda)$ is mildly homogeneous, for $\aleph_0 < \mathrm{cf}(\sigma) = \sigma < \lambda$,

(2) If P is mildly homogeneous and

$$\Vdash_P \text{“}\underset{\sim}{Q} \text{ is mildly homogeneous”},$$

<u>then</u> $P * \underset{\sim}{Q}$ is mildly homogeneous.

(3) $Q_{(\kappa,\lambda)}$ is mildly homogeneous, for $\aleph_0 < \mathrm{cf}(\kappa) = \kappa < \lambda$.

Proof of the Claim. (1) Suppose not, and let $p, q \in P \stackrel{\text{def}}{=} \mathbb{P}(\sigma,\lambda)$ force contradictory statements about $\varphi(a_0,\ldots,a_{n-1})$. Let $\alpha = \mathrm{Dom}(q)$ and consider the function $F : P \to P$ such that $F(f) = g$ iff $q \subseteq g$ and for $i \in \mathrm{dom}(f)$ we have $g(\alpha + i) = f(i)$.

This function is an isomorphism between P and $P/q \stackrel{\text{def}}{=} \{g \in P : g \supseteq q\}$, and it induces an isomorphism between P-names and P/q-names. However, in P/q we have that q and $F(p) \geq q$ force contradictory statements about $\varphi(a_0,\ldots,a_{n-1})$. Contradiction.

(2)-(3) Similar proofs. $\bigstar_{1.7}$

Remark 1.8. We remark that $\mathbb{P}(\sigma,\lambda)$ in fact has stronger homogeneity properties, a fact which will not be used here.

Proof of the Theorem continued.

Let V, $\mathbf{j}$ and the cardinals mentioned in the assumptions be fixed. Note that by elementarity, λ^* is the successor of a singular cardinal.

Notation 1.9. In the situation when notation $Q^{\otimes}_{(R,\kappa,\kappa^*)}$ makes sense, we abbreviate it as $Q^{\otimes}_{(R,\kappa)}$.

Definition 1.10. We define $\mathbb{P}^-$ to be the forcing whose elements are functions h, with $\mathrm{dom}(h)$ an Easton subset of μ_0, with the property

$$\alpha < \beta \in \mathrm{dom}(h) \Longrightarrow h(\alpha) \in \mathcal{H}(\beta),$$

ordered by extension.

Claim 1.11. Forcing with $\mathbb{P}^-$ preserves cardinals and cofinalities $\geq \mu_0$, and GCH above μ_0, while any inaccessible $\sigma \leq \mu_0$ which is a limit of inaccessibles, remains regular and $2^\sigma = \sigma^+$ holds in the extension by $\mathbb{P}^-$.

Proof of the Claim. First notice that $|\mathbb{P}^-| = \mu_0$, so $\mathbb{P}^-$ has μ_0^+-cc and preserves cardinals and cofinalities $\geq \mu_0^+$, as well as GCH above μ_0.

Now suppose that $\sigma \leq \mu_0$ is a limit of inaccessibles, but its cofinality is changed by $\mathbb{P}^-$ to be $\leq \theta$ for some $\theta < \sigma$. Let $p^* \in \mathbb{P}^-$ force this. Without loss of generality, θ is (strongly) inaccessible and $\theta \in \mathrm{dom}(p^*)$.

Let

$$P_{<\theta} \stackrel{\mathrm{def}}{=} \{q \restriction \theta : \, q \in \mathbb{P}^- \ \& \ q \geq p^*\}$$

and

$$P_{\geq\theta} \stackrel{\mathrm{def}}{=} \{q \restriction [\theta, \mu_0) : \, q \in \mathbb{P}^- \ \& \ q \geq p^*\},$$

both ordered by extension. The mapping $q \mapsto (q \restriction [\theta, \mu_0), q \restriction \theta)$ shows that $\mathbb{P}^-/p \stackrel{\mathrm{def}}{=} \{q \in \mathbb{P}^- : \, q \geq p^*\}$ is isomorphic to $P_{\geq\theta} \times P_{<\theta}$. We have that $P_{\geq\theta}$ is $(< \theta^+)$-closed, so $P_{<\theta}$ adds a cofinal function from θ to σ. However, $|P_{<\theta}| \leq \theta$ (as θ is strongly inaccessible), and so it preserves cardinals and cofinalities $\geq \theta^+$, a contradiction.

We can similarly decompose $\mathbb{P}^-$ into $P_{\geq\sigma} \times P_{<\sigma}$ to observe that

$$\Vdash_{\mathbb{P}^-} \text{“}2^\sigma = \sigma^+\text{”}.$$

$\bigstar_{1.11}$

Definition 1.12. (1) For $\mu < \mu_0$ let $\underset{\sim}{\mathbb{R}}_\mu$ and $\underset{\sim}{\kappa}_\mu$be the following $\mathbb{P}^-$-names: for a condition $p \in \mathbb{P}^-$, <u>if</u> $\mu \in \mathrm{dom}(p)$ and

$$p(\mu) = (\kappa, \underset{\sim}{R}) \text{ with } \mu < \mathrm{cf}(\kappa) = \kappa < \mu_0,$$

and $R \in \mathcal{H}(\kappa^+)$ is a forcing notion which preserves cardinals and cofinalities $\geq \mu$, <u>then</u> p forces $\underset{\sim}{\kappa}_\mu$ to be κ and $\underset{\sim}{\mathbb{R}}_\mu$ to be $\underset{\sim}{Q}^{\otimes}_{(R,\underset{\sim}{\kappa}_\mu)}$. We say that $R_\mu = R$. <u>If</u>, $\mu \in \mathrm{Dom}(p)$ but $p(\mu)$ does not satisfy the conditions above, <u>then</u> p forces $\underset{\sim}{\mathbb{R}}_\mu$ to be the trivial forcing, which will for notational purposes be thought of as $\{\emptyset, \emptyset)\}$. For the same reason, in these circumstances we think of $R_\mu = \{\emptyset\}$.

Note: each $\underset{\sim}{\mathbb{R}}_\mu$ is (over a dense subset of $\mathbb{P}^-$) a $\mathbb{P}^-$-name of a forcing notion from V, $\underset{\sim}{\kappa}_\mu$ is a $\mathbb{P}^-$-name of an ordinal $< \mu_0$, and $\prod_{\mu<\mu_0} \underset{\sim}{\mathbb{R}}_\mu$ is a $\mathbb{P}$-name of a product of forcing. But $\mathbb{R}$ below is forced not to be from V.

(2) For a $\mathbb{P}^-$-name $\underset{\sim}{f} \in \prod_{\mu<\mu_0} \underset{\sim}{\mathbb{R}}_\mu$ and $\alpha \leq \mu_0$, let

$$\underset{\sim}{A}_{\underset{\sim}{f},\alpha} \stackrel{\mathrm{def}}{=} \{\mu < \mu_0 : \underset{\sim}{f}(\mu) = (\emptyset, \underset{\sim}{q}) \text{ with } \neg(\Vdash_{\underset{\sim}{\mathbb{R}}_\mu} \text{“}\alpha \notin \mathrm{Dom}(\underset{\sim}{q})\text{”})\}.$$

(23) Let $\underset{\sim}{\mathbb{R}}$ be a $\mathbb{P}^-$-name for:

$$\left\{ \underset{\sim}{f} \in \prod_{\mu<\mu_0} \underset{\sim}{\mathbb{R}}_\mu : (\forall \alpha \leq \mu_0)\,[\underset{\sim}{A}_{\underset{\sim}{f},\alpha} \text{ is an Easton set }] \right\},$$

ordered by the order inherited from $\prod_{\mu<\mu_0} \underset{\sim}{\mathbb{R}}_\mu$.

hence, $\underset{\sim}{\mathbb{R}}$ is a $\mathbb{P}^-$-name of a forcing notion.

Notation 1.13. If we write $(p, \underset{\sim}{\bar{r}}) \in \mathbb{P}^- * \underset{\sim}{\mathbb{R}}$, we mean that

$$\Vdash_{\mathbb{P}^-} \underset{\sim}{\bar{r}} = \langle (\emptyset_{R_\mu}, \underset{\sim}{r}(\mu)) : \mu < \mu_0 \rangle.$$

Definition 1.14. (1) Given $(p, \underset{\sim}{\bar{r}}) \in \mathbb{P}^- * \underset{\sim}{\mathbb{R}}$ and $\sigma = \mathrm{cf}(\sigma) < \mu_0$. For $(q, \underset{\sim}{\bar{s}}) \in \mathbb{P}^- * \underset{\sim}{\mathbb{R}}$, we define

(i) $(q, \underset{\sim}{\bar{s}}) \geq_{\mathrm{pr},\sigma} (p, \underset{\sim}{\bar{r}})$ iff

(α) $(q, \underset{\sim}{\bar{s}}) \geq (p, \underset{\sim}{\bar{r}})$,

(β) $q \restriction (\sigma+1) = p \restriction (\sigma+1)$,

(γ) For $\mu < \mu_0$ with $\neg(q \Vdash$ "$\underset{\sim}{\mathbb{R}}_\mu$ is trivial"), we have

$$(q, \emptyset_{\underset{\sim}{R}_\mu}) \Vdash \text{"if } \underset{\sim}{\kappa}_\mu < \sigma, \text{ then } \underset{\sim}{s}(\mu) \restriction (\underset{\sim}{\kappa}_\mu, \sigma] = \underset{\sim}{r}(\mu) \restriction (\underset{\sim}{\kappa}_\mu, \sigma]\text{"}.$$

(ii) $(q, \underset{\sim}{\bar{s}}) \geq_{\mathrm{apr},\sigma} (p, \underset{\sim}{\bar{r}})$ iff

(α) $(q, \underset{\sim}{\bar{s}}) \geq (p, \underset{\sim}{\bar{r}})$,

(β) $\mathrm{dom}(q) \cap (\sigma+1, \mu_0) = \mathrm{dom}(p) \cap (\sigma+1, \mu_0)$.

(γ) For $\mu < \mu_0$ with $\neg(q \Vdash$ "$\underset{\sim}{\mathbb{R}}_\mu$ is trivial"), we have

$$(q, \emptyset_{\underset{\sim}{R}_\mu}) \Vdash \text{"}\underset{\sim}{s}(\mu) \restriction (\sigma, \kappa^*) = \underset{\sim}{r}(\mu) \restriction (\sigma, \kappa^*)\text{"}.$$

(2) For $(p, \underset{\sim}{\bar{r}}) \in \mathbb{P}^- * \underset{\sim}{\mathbb{R}}$ and $\sigma = \mathrm{cf}(\sigma) \leq \mu_0$, we let

$$Q^-_{(p,\underset{\sim}{\bar{r}}),\sigma} \stackrel{\mathrm{def}}{=} \{(q, \underset{\sim}{\bar{s}}) : (q, \underset{\sim}{\bar{s}}) \geq_{\mathrm{apr},\sigma} (p', \underset{\sim}{\bar{r}}') \text{ for some } (p', \underset{\sim}{\bar{r}}') \leq_{\mathrm{pr},\sigma} (p, \underset{\sim}{\bar{r}})\},$$

ordered as a suborder of $\mathbb{P}^- * \underset{\sim}{\mathbb{R}}$.

Claim 1.15. Given $(p, \underset{\sim}{\bar{r}}) \leq (q, \underset{\sim}{\bar{s}})$ in $\mathbb{P}^- * \underset{\sim}{\mathbb{R}}$, and a regular $\sigma < \mu_0$.

Then there is a unique $(t, \underset{\sim}{\bar{z}})$ such that

$$(p, \underset{\sim}{\bar{r}}) \leq_{\mathrm{pr},\sigma} (t, \underset{\sim}{\bar{z}}) \leq_{\mathrm{apr},\sigma} (q, \underset{\sim}{\bar{s}}).$$

Proof of the Claim. Let $t \stackrel{\mathrm{def}}{=} p \restriction (\sigma+1) \cup q \restriction (\sigma+1, \mu_0)$. Hence $t \in \mathbb{P}^-$ and $p \leq t \leq q$. We define a $\mathbb{P}^- * \underset{\sim}{\mathbb{R}}$-name $\underset{\sim}{\bar{z}}$ by letting for $\mu < \mu_0$

$$\underset{\sim}{z}(\mu) \stackrel{\mathrm{def}}{=} \begin{cases} \underset{\sim}{r}(\mu) \restriction (\underset{\sim}{\kappa}_\mu, \sigma] \frown \underset{\sim}{q}(\mu) \restriction (\sigma, \kappa^*] & \text{if defined} \\ \underset{\sim}{r}(\mu) & \text{otherwise.} \end{cases}$$

$\bigstar_{1.15}$

Notation 1.16. $(t, \underset{\sim}{\bar{z}})$ as in Claim 1.15 is denoted by $\mathrm{intr}((p, \underset{\sim}{\bar{r}}), (q, \underset{\sim}{\bar{s}}))$.

Claim 1.17. For $\sigma = \mathrm{cf}(\sigma) < \mu_0$, the forcing $(\mathbb{P}^- * \underset{\sim}{\mathbb{R}}, \leq_{\mathrm{pr},\sigma})$ is $(< \sigma + 1)$-strategically closed.

Proof of the Claim. For every $\mu < \mu_0$ we have

$\Vdash_{\mathbb{P}^- * \underset{\sim}{\mathbb{R}}}$ "$\underset{\sim}{\mathbb{R}}_\mu$ non-trivial $\Longrightarrow \underset{\sim}{Q}_{(\underset{\sim}{\kappa}_\mu, \kappa^*)} / \underset{\sim}{Q}_{(\underset{\sim}{\kappa}_\mu, \sigma]}$ is $(< \sigma + 1)$-strategically closed."

Hence we can find names $\underset{\sim}{\mathrm{St}}^\sigma_\mu$ of the corresponding winning strategies which exemplify the above statement.

Suppose that $\zeta \leq \sigma$ and $\langle p_\xi = \langle p^0_\xi, \underset{\sim}{\bar{p}}^1_\xi \rangle : \xi < \zeta \rangle$, $\langle q_\xi = \langle q^0_\xi, \underset{\sim}{\bar{q}}^1_\xi \rangle : \xi < \zeta \rangle$ are sequences of elements of $\mathbb{P}^- * \underset{\sim}{\mathbb{R}}$ such that

(1) For all $\xi < \zeta$ we have $p_\xi \leq_{\mathrm{pr},\sigma} q_\xi$,

(2) For all $\xi < \zeta$ and $\varepsilon < \xi$ we have $q_\varepsilon \leq_{\mathrm{pr},\sigma} p_\xi$ and for $\mu < \mu_0$ with $\neg(p_\xi \Vdash$ "$\underset{\sim}{\mathbb{R}}_\mu$ is trivial"), we have $(p_\xi, \emptyset_{\underset{\sim}{R}_\mu}, \underset{\sim}{p}^1_0(\mu) \restriction (\underset{\sim}{\kappa}_\mu, \sigma]) \Vdash_{\mathbb{P}^- * \underset{\sim}{\mathbb{R}}_\mu * \underset{\sim}{Q}_{(\kappa_\mu, \sigma]}}$

"$\underset{\sim}{p}^1_\xi(\mu) \restriction (\sigma, \kappa^*) = \underset{\sim}{\mathrm{St}}^\sigma_\mu(\langle \underset{\sim}{p}^1_\varepsilon(\mu) \restriction (\sigma, \kappa^*) : \varepsilon < \xi \rangle, \langle \underset{\sim}{q}^1_\varepsilon(\mu) \restriction (\sigma, \kappa^*) : \varepsilon < \xi \rangle)$"

We define p_ζ by letting $p^0_\zeta \stackrel{\mathrm{def}}{=} \bigcup \{q^0_\xi : \xi < \zeta\}$. Notice that $p^0_\zeta \in \mathbb{P}^-$ and $p^0_\zeta \restriction (\sigma + 1) = p^0_0 \restriction (\sigma + 1)$.

For $\mu < \mu_0$ with$\neg(p_\xi \Vdash$ "$\underset{\sim}{\mathbb{R}}_\mu$ is trivial"), we let $\underset{\sim}{p}^1_\zeta(\mu)$ be the name given by

$$\underset{\sim}{p}^1_\zeta(\mu) \restriction (\underset{\sim}{\kappa}_\mu, \sigma] \stackrel{\mathrm{def}}{=} \underset{\sim}{p}^0_\zeta(\mu) \restriction (\underset{\sim}{\kappa}_\mu, \sigma]$$

and

$$\underset{\sim}{p}^1_\zeta(\mu) \restriction (\sigma, \kappa^*) \stackrel{\mathrm{def}}{=} \underset{\sim}{\mathrm{St}}^\sigma_\mu(\langle \underset{\sim}{p}^1_\xi(\mu) \restriction (\sigma, \kappa^*) : \xi < \zeta \rangle, \langle \underset{\sim}{q}^1_\xi(\mu) \restriction (\sigma, \kappa^*) : \xi < \zeta \rangle).$$

$\bigstar_{1.17}$

Claim 1.18. Suppose $(p, \underset{\sim}{\bar{r}}) \Vdash$ "$\underset{\sim}{\tau} : \sigma \to \mathrm{Ord}$", where σ is regular $< \mu_0$.

Then there is $(q, \underset{\sim}{\bar{s}}) \geq_{\mathrm{pr},\sigma} (p, \underset{\sim}{\bar{r}})$ and a $Q^-_{(q, \underset{\sim}{\bar{s}}), \sigma}$-name $\underset{\sim}{\tau}'$ such that

$$(q, \underset{\sim}{\bar{s}}) \Vdash \text{"}\underset{\sim}{\tau} = \underset{\sim}{\tau}'\text{"}.$$

Proof of the Claim. We define a play of $G((\mathbb{P}^- * \underset{\sim}{\mathbb{R}}, \leq_{\mathrm{pr},\sigma}), \sigma)$ as follows.

I starts by playing $(p, \underset{\sim}{\bar{r}}) \stackrel{\mathrm{def}}{=} p_0$. At the stage $\zeta \leq \sigma$, player II chooses $q^*_\zeta \geq p_\zeta$ such that q^*_ζ forces a value to $\underset{\sim}{\tau}_\zeta$, and we let $q_\zeta \stackrel{\mathrm{def}}{=} \mathrm{intr}(p_\zeta, q^*_\zeta)$. At the stage $0 < \zeta < \sigma$, we let I play according to the winning strategy for $G((\mathbb{P}^- * \underset{\sim}{\mathbb{R}}, \leq_{\mathrm{pr},\sigma}), \sigma)$ applied to $(\langle p_\xi : \xi < \zeta \rangle, \langle q_\xi : \xi < \zeta \rangle)$. At the end, we let $(q, \underset{\sim}{\bar{s}}) = p_\sigma$. $\bigstar_{1.18}$

Claim 1.19. If $(p, \underset{\sim}{\bar{r}}) \in \mathbb{P}^- * \underset{\sim}{\mathbb{R}}$, and $\sigma = \mathrm{cf}(\sigma) < \mu_0$ is such that $\sigma \in \mathrm{dom}(p)$, <u>then</u> $Q^-_{(p,\underset{\sim}{\bar{r}}),\sigma}$ satisfies μ_0-cc.

Proof of the Claim. Given $\bar{q} = \langle q_i = \langle q^0_i, \underset{\sim}{q}^1_i \rangle : i < \mu_0 \rangle$, with $q_i \in Q^-_{(p,\underset{\sim}{\bar{r}}),\sigma}$. Suppose for contradiction that the range of this sequence is an antichain.

We have that for all $i < \mu_0$

$$q^0_i \restriction (\sigma + 1, \mu_0) \subseteq p \restriction (\sigma + 1, \mu_0).$$

As $\mathrm{dom}(q^0_i)$ is an Easton set, without loss of generality we have that all $q^0_i \restriction (\sigma + 1, \mu_0)$ are the same q^*. Let G^- be $\mathbb{P}^-$-generic over V with $q^* \in G^-$. Hence in $V[G^-]$ the sequence $\langle q^1_i \stackrel{\mathrm{def}}{=} (\emptyset, \underset{\sim}{\bar{q}}^i) : i < \mu_0 \rangle$ to an antichain in $\prod_{\mu < \mu_0} [\mathbb{R}_\mu * \underset{\sim}{Q}_{(\kappa_\mu, \kappa^*)}]^\otimes$, and by the choice of the initial sequence, we have that $\langle (\emptyset, \underset{\sim}{q}^i(\mu) \restriction (\sigma + 1)) : \mu < \mu_0 \rangle : i < \mu_0 \rangle$ gives rise to an antichain in $\prod_{\mu < \mu_0} [\mathbb{R}_\mu * \underset{\sim}{Q}_{\kappa_\mu, \sigma}]^\otimes$. For every $i < \mu_0$,

$$A_i \stackrel{\mathrm{def}}{=} \{\mu < \mu_0 : \underset{\sim}{q}^i_\mu \restriction (\sigma + 1) \neq \emptyset\}$$

has size $\leq \sigma$. Hence, without loss of generality, A_i's form a Δ-system with root A^*. Hence

$$\Big\langle \langle \langle \emptyset, \underset{\sim}{q}^i(\mu) \restriction (\sigma + 1) \rangle : \mu \in A^* : \rangle\, i < \mu_0 \Big\rangle$$

gives rise to an antichain, a contradiction. $\bigstar_{1.19}$

Claim 1.20. Forcing with $\mathbb{P}^- * \underset{\sim}{\mathbb{R}}$ preserves cardinals and cofinalities $\geq \mu_0$.

Proof of the Claim. Suppose cofinalities $\geq \mu_0$ are not preserved and let θ be the first cofinality $\geq \mu_0$ destroyed. Hence θ is regular, and for some $\underset{\sim}{\tau}$, condition $(p, \bar{\underset{\sim}{r}})$ and regular $\sigma < \theta$, we have $(p, \bar{\underset{\sim}{r}}) \Vdash$ "$\underset{\sim}{\tau} : \sigma \to \theta$ is cofinal".

Case 1. $\sigma < \mu_0$. By Claim 1.18, there is $(q, \bar{\underset{\sim}{s}}) \geq (p, \bar{\underset{\sim}{r}})$ and a $Q^-_{(p,\bar{\underset{\sim}{r}}),\sigma}$-name $\underset{\sim}{\tau}'$ such that $(q, \bar{\underset{\sim}{s}}) \Vdash$ "$\underset{\sim}{\tau} = \underset{\sim}{\tau}'$". Hence $(q, \bar{\underset{\sim}{s}}) \Vdash$ "$\underset{\sim}{\tau}' : \sigma \to \theta$ is cofinal", contradicting the fact that $Q^-_{(p,\bar{\underset{\sim}{r}}),\sigma}$ has μ_0-cc.

Case 2. $\sigma \geq \mu_0$.

As for every $\mu < \mu_0$ with $\underset{\sim}{\mathbb{R}}_\mu$ non-trivial we have that

$$\mathbb{P}^- * \underset{\sim}{\mathbb{R}}_\mu * \underset{\sim}{Q}_{(\underset{\sim}{\kappa}_\mu,\kappa^*)}/\underset{\sim}{Q}_{(\underset{\sim}{\kappa}_\mu,\sigma]} \text{ is } (<\sigma+1)\text{-strategically closed,}$$

there is $(q, \bar{\underset{\sim}{s}}) \geq (p, \bar{\underset{\sim}{r}})$ and a $\mathbb{P}^- * \prod_{\mu<\mu_0}[\underset{\sim}{\mathbb{R}}_\mu * \underset{\sim}{Q}_{(\underset{\sim}{\kappa}_\mu,\sigma]}]^{\otimes}$-name $\underset{\sim}{\tau}'$ such that $(q, \bar{\underset{\sim}{s}}) \Vdash$ "$\underset{\sim}{\tau}' : \sigma \to \theta$ is cofinal". But this forcing has σ^+-cc, a contradiction. $\bigstar_{1.20}$

Corollary 1.21. Forcing with $\mathbb{P}^- * \underset{\sim}{\mathbb{R}} * \underset{\sim}{Q}_{(\kappa^*,\lambda^*)}$ preserves cardinalities and cofinalities $\geq \mu_0$, and forces $SNR(\theta, \kappa^*)$ for $\theta \in (\kappa^*, \lambda^*)$.

Claim 1.22. The following is forced by $\mathbb{P}^-$:

(1) $\underset{\sim}{\mathbb{R}}$ is mildly homogeneous.

(2) $\underset{\sim}{\mathbb{R}} * \underset{\sim}{Q}_{(\kappa^*,\lambda^*)}$ is mildly homogeneous.

Proof of the Claim. (1) First note that each $\underset{\sim}{\mathbb{R}}_\mu$ is forced to be mildly homogeneous, by Claim 1.7(2) and the definition of $\otimes$ operation.

(2) Follows from (1) and Claim 1.7(2). $\bigstar_{1.22}$

Main Claim 1.23. After forcing with $\mathbb{P} \overset{\text{def}}{=} \mathbb{P}^- * \underset{\sim}{\mathbb{R}} * \underset{\sim}{Q}_{(\kappa^*,\lambda^*)}$, we have the weak reflection of λ^* at κ^*.

Proof of the Main Claim. Suppose otherwise, and let $p^* = (p, \underset{\sim}{q}, \underset{\sim}{r})$ force $\underset{\sim}{\tau}$ to be a function exemplifying the strong non-reflection of λ^* at κ^*. As $\underset{\sim}{\mathbb{R}} * \underset{\sim}{Q}_{(\kappa^*,\lambda^*)}$ is forced to be mildly homogeneous by Claim 1.22, without loss of generality $p^* = (p, \emptyset, \emptyset)$.

By a standard argument, our large cardinal assumptions imply that we can find a model $N \prec \mathcal{H}(\chi)$ such that

(i) $N \cap \mu_0$ is an ordinal $\mu < \mu_0$,

(ii) $\mathrm{otp}(N \cap \lambda^*) = \kappa^*$,

(iii) $\mathrm{otp}(N \cap \mu_1) = \mu_0$,

(iv) ${}^{\omega}N \subseteq N$ (even ${}^{\mu>}N \subseteq N$, although we do not use this),

(v) $(N, \in)$ is isomorphic to $\mathcal{H}(\chi')$ for some regular $\chi' < \chi$

(vi) $|N \cap \kappa^*|$ is a regular cardinal.

(vii) $\kappa^*, \mu_0, \mu_1, \mu_2, \lambda^*, \mathbb{P}, p^*, \underset{\sim}{\tau} \in N$.

[Why? Consider $\mathbf{j}$“$(\mathcal{H}(\chi))$ and use elementarity. Note that by ${}^{\chi}M \subseteq M$ and $\chi^{<\chi} = \chi$ we have that $\mathbf{j}$“$(\mathcal{H}(\chi)) \in M$.]

First we consider some consequences of our choice of N. Let $\kappa \stackrel{\mathrm{def}}{=} |N \cap \kappa^*|$ and $\delta \stackrel{\mathrm{def}}{=} \sup(N \cap \lambda^*)$.

Our assumptions on N imply that $\mathrm{otp}(N \cap \kappa^*) < \mu_0$, hence $\kappa < \mu_0$. As for δ, we have $\delta \in S^{\lambda^*}_{\kappa^*}$. Now notice that $N \cap \delta$ is stationary in δ, and it remains so after forcing with $\mathbb{P}$.

[Why? The set $S \stackrel{\mathrm{def}}{=} S^{\delta}_{\aleph_0} \cap N$ is a stationary subset of δ, as E defined as the closure of $N \cap \delta$ is a club of δ, and $[\alpha \in N \ \&\ \mathrm{cf}(\alpha) = \aleph_0] \Longrightarrow \alpha \in S$ (this is true even with“$\mathrm{cf}(\alpha) < \mu$” in place of “$\mathrm{cf}(\alpha) = \aleph_0$”). But $\mathbb{P}$ is an ω_1-closed forcing notion, hence S remains stationary after forcing with $\mathbb{P}$.]

As $p^* \in N$, we have that $\mathrm{dom}(p) \subseteq \mu$. Hence

$$p^+ \stackrel{\mathrm{def}}{=} p \cup \{\langle \mu, (\kappa, (\mathbb{P}^- * \underset{\sim}{\mathbb{R}})^N)\rangle\}$$

is a well defined condition in $\mathbb{P}^-$, and it extends p. In fact, p^+ is a $\mathbb{P}^-$-generic condition over N, and it forces that $\underset{\sim}{\mathbb{R}}_\mu$ is $(\mathbb{P}^- * \underset{\sim}{\mathbb{R}})^N$, hence that $[\underset{\sim}{\mathbb{R}}_\mu * \underset{\sim}{Q}_{(\kappa,\kappa^*)}]^{\otimes}$ is $[(\mathbb{P}^- * \underset{\sim}{\mathbb{R}})^N * \underset{\sim}{Q}_{(\kappa,\kappa^*)}]^{\otimes}$, which is $([(\mathbb{P}^- * \underset{\sim}{\mathbb{R}}) * \underset{\sim}{Q}_{(\kappa^*,\lambda^*)}]^{\otimes})^N$. Let $\underset{\sim}{H}$ be $([(\mathbb{P}^- * \underset{\sim}{\mathbb{R}}) * \underset{\sim}{Q}_{(\kappa^*,\lambda^*)}]^{\otimes})^N$-generic with $p \in \underset{\sim}{H}$. The inverse of the Mostowski collapse F of N maps $\underset{\sim}{H}$ into a subset $\underset{\sim}{H}^*$ of $[(\mathbb{P}^- * \underset{\sim}{\mathbb{R}}) * \underset{\sim}{Q}_{(\kappa^*,\lambda^*)}]^{\otimes}$. Notice then that

$$p^+ \Vdash \text{“}\underset{\sim}{H} \text{ is } \underset{\sim}{\mathbb{R}}_\mu\text{-generic”}.$$

We wish to define q as follows: $q \stackrel{\text{def}}{=} (p^+, \emptyset_{\mathbb{R}}, \underset{\sim}{r})$, where $\underset{\sim}{r}$ is a $\mathbb{P}^- * \underset{\sim}{\mathbb{R}}$-name over $(p^+, \emptyset_{\underset{\sim}{\mathbb{R}}})$ of a condition in $\underset{\sim}{Q}_{(\kappa^*, \lambda^*)}$ defined by letting

$$\mathrm{Dom}(\underset{\sim}{r}) \stackrel{\text{def}}{=} \bigcup \{\mathrm{Dom}(\underset{\sim}{h}) : (p^+, \emptyset_{\underset{\sim}{\mathbb{R}}}) \Vdash \text{“}F^{-1}(\underset{\sim}{h}) \in \underset{\sim}{H}\text{”}\},$$

and for θ with $(p^+, \emptyset_{\underset{\sim}{\mathbb{R}}}) \Vdash$ “$\theta \in \mathrm{Dom}(\underset{\sim}{r})$”, we let

$$\underset{\sim}{r}(\theta) \stackrel{\text{def}}{=} \bigcup \{\underset{\sim}{h}(\theta) : (p^+, \emptyset_{\underset{\sim}{\mathbb{R}}}) \Vdash \text{“}\theta \in \mathrm{Dom}(\underset{\sim}{h}) \ \& \ \underset{\sim}{h} \in \underset{\sim}{H}^*\text{”}\}.$$

We now claim that q is a condition in $\mathbb{P}$ and $q \geq p^*$. Let us check the relevant items:

(a) If $(p^+, \emptyset) \Vdash$ “θ strongly inaccessible $\in (\kappa^*, \lambda^*)$”, then

$$(p^+, \emptyset) \Vdash \text{“}|\mathrm{Dom}(\underset{\sim}{r}) \cap \theta| < \theta\text{”}.$$

[Why? We have that for some $\theta' \in (\kappa, \kappa^*)$,

$$(p^+, \emptyset) \Vdash \text{“}\mathrm{Dom}(\underset{\sim}{r}) \cap \theta \subseteq \bigcup \{\mathrm{Dom}(F(\underset{\sim}{f}) : \underset{\sim}{f} \in \underset{\sim}{Q}^N_{(\kappa, \theta')}\}\text{”}.$$

But then, $(p^+, \emptyset) \Vdash$ “$|\underset{\sim}{Q}^N_{(\kappa,\theta')}| \leq |2^{\theta'} \cap N| < \kappa^*$”.]

(b) If $(p^+, \emptyset) \Vdash$ “$\theta \in \mathrm{Dom}(\underset{\sim}{r})$”, then $(p^+, \emptyset) \Vdash$ “$\underset{\sim}{r}(\theta)$ is a function whose domain is an ordinal $< \theta$ and range a subset of κ^*”.

[Why? As $(p^+, \emptyset) \Vdash$ “$\underset{\sim}{H}^*$ is directed”, we have that $(p^+, \emptyset) \Vdash$ “$\underset{\sim}{r}(\theta)$ is a function”. If

$$(p^+, \emptyset) \Vdash \text{“}\theta \in \mathrm{Dom}(\underset{\sim}{h}) \ \& \ F^{-1}(\underset{\sim}{h}) \in \underset{\sim}{H}\text{”},$$

then $(p^+, \emptyset)$ forces

$$\text{“}(\forall \sigma \in \mathrm{Dom}(F^{-1}(\underset{\sim}{h})))\, [F^{-1}(\underset{\sim}{h})(\sigma) \text{ is a function with domain } \in \sigma]\text{”},$$

so by elementarity

$$(p^+, \emptyset) \Vdash \text{“}\mathrm{dom}(\underset{\sim}{h}(\theta)) \text{ is an element of } \theta\text{”}.]$$

(c) $(p^+, \emptyset) \Vdash$ “$\underset{\sim}{r} \in \underset{\sim}{Q}_{(\kappa^*, \lambda^*)}$.

[Why? First of all, we know that $(p^+, \emptyset) \Vdash$ "$\mathrm{dom}(\underset{\sim}{r}) \subseteq \lambda^*$ with no last element". Now, for relevant θ, $\mathrm{dom}(\underset{\sim}{r}(\theta))$ is the union of a subset of $\mathcal{H}(\theta^{++}) \cap N$ which has cardinality $\leq |\theta^{++} \cap N| < \kappa^*$. Hence the union has cofinality $< \kappa^*$ (as having cofinality $\geq \kappa^*$ is preserved by the forcing), hence $\underset{\sim}{r}(\theta)$ is forced to be in $\underset{\sim}{\mathbb{P}}(\theta, \kappa^*)$, and hence $\underset{\sim}{r}$ is forced to be an element of $\underset{\sim}{Q}_{(\kappa^*, \lambda^*)}$.]

(d) $(p^+, \emptyset)$ forces that for all $\varepsilon \in S^{\theta}_{\kappa^*}$, there is a club e of ε on which $\underset{\sim}{r}(\theta)$ is strictly increasing, for $\theta \in \mathrm{dom}(\underset{\sim}{r})$.

[Why? Because this is forced about each $\underset{\sim}{h}(\theta)$ for $\underset{\sim}{h} \in \underset{\sim}{H}^*$ and there are $< \kappa^*$ such $\underset{\sim}{h}$.]

But now we shall see that q forces $\underset{\sim}{\tau}$ to be constant on a stationary subset of δ, a contradiction, as $\delta \in S^{\lambda^*}_{\kappa^*}$, and remains there after forcing with $\mathbb{P}$. We need to consider what q forces about $\underset{\sim}{\tau}(\alpha)$ for $\alpha \in N$. Such $\underset{\sim}{\tau}(\alpha)$ is a $\mathbb{P}$-name of an ordinal $< \kappa^*$. Let

$$\mathcal{I}_\alpha \stackrel{\text{def}}{=} \{(\emptyset, \emptyset, \underset{\sim}{t}) \in \mathbb{P} : (\emptyset, \emptyset, \underset{\sim}{t}) \text{ forces } \underset{\sim}{\tau}(\alpha) \text{ to be equal to a } \mathbb{P}^- * \underset{\sim}{\mathbb{R}}\text{-name}\}.$$

Hence $\mathcal{I}_\alpha \in N$. As $\mathbb{P}^- * \underset{\sim}{\mathbb{R}}$ forces that $\underset{\sim}{Q}_{(\kappa^*, \lambda^*)}$ is $(\kappa^* + 1)$-strategically closed, we have that $\mathcal{I}_\alpha$ is dense in $[(\mathbb{P}^- * \underset{\sim}{\mathbb{R}}) * \underset{\sim}{Q}_{(\kappa^*, \lambda^*)}]^{\otimes}$. By the definition of $\underset{\sim}{H}^*$, there is $(\emptyset, \emptyset, \underset{\sim}{h}) \in \mathcal{I}_\alpha \cap N$ such that $(\emptyset, \emptyset, \underset{\sim}{h}) \leq q$. Let $\underset{\sim}{\tau}'$ exemplify this, so $\underset{\sim}{\tau}' \in N$.

Hence q forces $\underset{\sim}{\tau}(\alpha)$ to be in the set of all $\underset{\sim}{\tau}' \in N$, where $\underset{\sim}{\tau}'$ is a $\mathbb{P}^- * \underset{\sim}{\mathbb{R}}$-name of an ordinal $< \kappa^*$. The cardinality of this set $\leq |\mathcal{P}(\kappa^*) \cap N|$, which is $< \mu_0$. Since $\alpha \in N$ was arbitrary, q forces the range of $\underset{\sim}{\tau} \restriction (N \cap \delta)$ to be a set of size $< \mu_0$, hence $\underset{\sim}{\tau}$ will be constant on a stationary subset of $N \cap \delta$ (as $N \cap \delta$ is stationary). $\bigstar_{1.23}$

Proof of the Theorem continued.

(2) Same proof.

(3) Follow the forcing from (1) by a Levy collapse. We are making use of the following

Claim 1.24. Suppose λ weakly reflects at κ and P is a κ-cc forcing.

Then λ weakly reflects at κ in V^P.

Proof of the Claim. Suppose that

$$p \Vdash_P \text{“}\underset{\sim}{f} : \lambda \to \kappa\text{”}.$$

We define $f' : \lambda \to \kappa$ by letting $f'(\alpha) \stackrel{\text{def}}{=} \sup\{\gamma < \kappa : \neg(p \Vdash \text{“}\underset{\sim}{f}(\alpha) \neq \gamma\text{”})\}$. As P is κ-cc, the range of f' is indeed contained in κ. Let $\delta \in S^\lambda_\kappa$ be such that $f' \restriction S$ is constant on a stationary set $S \subseteq \delta$ (the existence of such δ follows as $WR(\lambda, \kappa)$ holds). Hence $p \Vdash$ “$\underset{\sim}{f} \restriction S$ is bounded”, so $\underset{\sim}{f}$ does not witness $SNR(\lambda, \kappa)$ in V^P, as S remains stationary in V^P. $\bigstar_{1.24}$

So, for example to get $\kappa^* = \aleph_n$ for $n \geq 1$, we could in $V^{\mathbb{P}}$ from (1) first make GCH hold below μ_0 by collapsing various cardinals below μ_0, and then collapse κ^* to be $\aleph_n$.

If we start in V by having $\kappa^* = \mu_0^{\omega+1}$, and in $V^{\mathbb{P}}$ collapse μ_0 to $\aleph_1$, then we get $\kappa^* = \aleph_{\omega+1}$.

For the last statement, start by $\kappa^* = \mu_0^{+\delta}$ for some $\delta \leq \aleph_7$, so $\lambda^* = \mu_1^{+\delta}$. By a minor change in the definition of $\mathbb{P}^-$, we can make $\mathbb{P}$ not add any $\aleph_7$-sequences from V. Now in $V^{\mathbb{P}}$, collapse first μ_0 to $\aleph_7$, and then collapse μ_1 to $(\kappa^*)^{+3}$. Hence $\lambda^* = (\kappa^*)^{+3+\delta}$. $\bigstar_{0.2}$

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